

Chapter 1

Nonlinear left and right eigenvectors for max-preserving maps

Björn S. Rüffer

Abstract It is shown that max-preserving maps (or join-morphisms) on the positive orthant in Euclidean n -space endowed with the component-wise partial order give rise to a semiring. This semiring admits a closure operation for maps that generate stable dynamical systems. For these monotone maps, the closure is used to define suitable notions of left and right eigenvectors that are characterized by inequalities. Some explicit examples are given and applications in the construction of Lyapunov functions are described.

1.1 Introduction

Classical Perron-Frobenius theory asserts the existence of nonnegative left and right eigenvectors corresponding to the dominant eigenvalue of a non-negative matrix [3, 4, 5, 9, 10]. For (nonlinear) monotone mappings from a positive cone into itself, various extensions to this theory have been developed, see [8] and the references therein. While most of the nonlinear extensions consider some form of right eigenvalue problem for monotone cone mappings, the question of left eigenvectors has not found a lot of attention. One reason that left eigenvectors do not have obvious counterparts in the world of nonlinear mappings may be that they are naturally elements of the (linear) dual of the underlying vector space in the classical spectral theory of linear operators. Linear duals are not very natural places to look for nonlinear eigenvectors.

In this contribution we consider a class of monotone mappings defined on the positive cone in \mathbb{R}^n equipped with the component-wise partial order. It admits a suitable notion of left eigenvectors. This class consists of *max-preserving* mappings from \mathbb{R}_+^n into itself, i.e., continuous, monotone maps

Björn S. Rüffer

School of Mathematical and Physical Sciences, The University of Newcastle (UON),
Callaghan, NSW 2308, Australia, e-mail: bjorn.ruffer@newcastle.edu.au

$A: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ with $A0 = 0$ for which $\max\{Ax, Ay\} = A \max\{x, y\}$. Instead of a numerical maximal eigenvalue, we consider the case when a nonlinear extension of the spectral radius is less than one, which can be characterised by the requirement that $A^k x \rightarrow 0$ as $k \rightarrow \infty$ for any $x \in \mathbb{R}_+^n$, or alternatively by the inequality

$$Ax \not\geq x \text{ for all } x \in \mathbb{R}_+^n, x \neq 0.$$

Given this starting point, it is not surprising that our nonlinear left and right “eigenvectors” are characterised by inequalities rather than equations. The terms “sub-eigenvectors” and spectral inequalities have been suggested as alternative terms for the objects introduced here. Both are (nonlinear) functions $l: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ and $r: \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ that are continuous, zero at zero, monotone and unbounded in every component. They satisfy

$$l(Ax) < l(x)$$

for all $x \in \mathbb{R}_+^n$, $x > 0$ as well as

$$A(r(t)) < r(t)$$

for all $t > 0$.

Both, l and r are defined via the closure A^* of A in the semiring of max-preserving maps on \mathbb{R}_+^n .

This contribution is organised as follows. The next section provides a little more background on our interest in left eigenvectors. In Section 1.3 we recall some necessary notation and preliminary results. Section 1.4 contains our main results with formulas for left and right eigenvectors in Theorems 2 and 3, respectively. Two explicit examples are given in Section 1.5. In Section 1.6 we explain how these eigenvectors can be used to construct Lyapunov functions. Section 1.7 concludes this contribution.

1.2 Motivation

Our interest in left eigenvectors is rooted in the stability analysis of interconnected systems, where the construction of Lyapunov functions for monotone comparison systems is of special interest [2].

For a dynamical system $x(k+1) = Ax(k)$, evolving on \mathbb{R}_+^n , a Lyapunov function $V: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is an energy function that decreases along trajectories. Lyapunov functions are used to prove that trajectories converge to zero, to prove stability, or to compute regions of attraction. Finding Lyapunov functions, however, is notoriously hard. Basic properties they need to satisfy are continuity, positive definiteness, radial unboundedness (i.e., $\|x\| \rightarrow \infty$ implies $V(x) \rightarrow \infty$) and descent along trajectories, i.e., $V(Ax) < V(x)$ whenever $x \neq 0$.

If $A \in \mathbb{R}_+^{n \times n}$ has spectral radius less than one, one can find a positive vector r (even in the case that A is merely nonnegative [11, Lemma 1.1]) so that $Ar \ll r$, i.e., the image under A of r is less than the vector r in every component. Such a vector determines a Lyapunov function via $V(x) = \max_i x_i/r_i$, and this Lyapunov function is called max-separable.

Max-separable Lyapunov functions exist for various monotone but nonlinear systems as well, but not for all [2]. In some of these nonlinear cases one can instead find sum-separable Lyapunov functions, which are of the form $V(x) = \sum_i v_i(x_i)$. If again $A \in \mathbb{R}_+^{n \times n}$ has spectral radius less than one, i.e., in the linear case, there exists a positive vector $l \in \mathbb{R}_+^n$, so that $l^T A \ll l^T$. This vector, too, determines a Lyapunov function, $V(x) = l^T x$, and this one is sum-separable. For general monotone systems however, these sum-separable Lyapunov functions are not well understood yet, although progress has been made in some special cases [2, 6].

As left Perron eigenvectors do determine (sum-) separable Lyapunov functions in the linear case, there is hope that a suitable notion of left eigenvectors will also provide Lyapunov functions in more general scenarios. It turns out, however, that while the present definition of left-eigenvectors does yield Lyapunov functions given by explicit formulas, these Lyapunov functions are not separable in the above sense.

1.3 Preliminaries

In this work we consider \mathbb{R}^n equipped with the component-wise partial order, which generates the positive cone $\mathbb{R}_+^n = [0, \infty)^n$. We use the following notation.

$$\begin{aligned} x &\leq y \text{ if } y - x \in \mathbb{R}_+^n, \\ x &< y \text{ if } x \leq y \text{ and } x \neq y, \\ x &\ll y \text{ if } y - x \text{ are in the interior of } \mathbb{R}_+^n. \end{aligned}$$

Note that $\max\{x, y\}$ is the component-wise maximum of the two vectors $x, y \in \mathbb{R}^n$. For notational convenience we use the binary symbol $x \oplus y$ to denote the same thing. We also write $\bigoplus\{x_k\}$ to denote the component-wise supremum of a possibly infinite set $\{x_k\}$ of vectors $x_k \in \mathbb{R}^n$.

By $\|x\| = \max_i |x_i|$ we denote the maximum-norm of $x \in \mathbb{R}^n$. We note that for $x, y \in \mathbb{R}^n$ we have $\|x \oplus y\| \leq \max\{\|x\|, \|y\|\}$ and equality holds if $x, y \in \mathbb{R}_+^n$.

The vector $(1, \dots, 1)^T \in \mathbb{R}^n$ will be denoted by $\mathbf{1}$. The standard unit vectors in \mathbb{R}^n are denoted by e_1, \dots, e_n .

In this work we will restrict our attention to continuous and monotone mappings. A mapping A is monotone if it preserves the partial order, i.e., $Ax \leq Ay$ whenever $x \leq y$. The set of *max-preserving* mappings from \mathbb{R}_+^n into

itself is given by

$$\text{MP} = \text{MP}(\mathbb{R}_+^n) = \left\{ A: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \text{ such that } \right. \\ \left. A(x \oplus y) = (Ax) \oplus (Ay) \text{ for all } x, y \in \mathbb{R}_+^n \right\}.$$

The term max-preserving map has been coined in [7] in the context of stability analysis of interconnected control systems. It coincides with the notion of join-morphisms in lattice theory [1]. It is immediate that max-preserving mappings are also monotone.

For $A \in \text{MP}$ we define non-decreasing functions $a_{ij}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $i, j = 1, \dots, n$, by $a_{ij}(t) = (A(te_j))_i$ for $t \in \mathbb{R}_+$. It is immediate that A can be represented as

$$Ax = \begin{pmatrix} a_{11}(x_1) \oplus \dots \oplus a_{1n}(x_n) \\ \vdots \\ a_{n1}(x_1) \oplus \dots \oplus a_{nn}(x_n) \end{pmatrix},$$

so it is natural to think of A as the matrix (a_{ij}) .

We state the following observation, where \circ refers to composition.

Lemma 1. *The set MP is a (\circ, \oplus) -semiring with identity element $\text{id}_{\mathbb{R}_+^n}$ and neutral element $0_{\mathbb{R}_+^n}$.*

Proof. If $A, B \in \text{MP}$ then we verify

$$(A \circ B)(x \oplus y) = A(Bx \oplus By) = (A \circ B)x \oplus (A \circ B)y,$$

so MP is closed under composition and

$$\begin{aligned} (A \oplus B)(x \oplus y) &= A(x \oplus y) \oplus B(x \oplus y) = \\ &= (Ax \oplus Ay) \oplus (Bx \oplus By) = (Ax \oplus Bx) \oplus (Ay \oplus By) = \\ &= (A \oplus B)x \oplus (A \oplus B)y, \end{aligned}$$

so MP is closed under the maximum operation as well.

Clearly the identity $\text{id}_{\mathbb{R}_+^n}$ is a member of MP and it is the identity element for composition. The function $0 = 0_{\mathbb{R}_+^n}$, which sends all of \mathbb{R}_+^n to $0 \in \mathbb{R}_+^n$, is in MP , and it serves as neutral element for the maximum operation. $\$$

For convenience we will write compositions simply as products, i.e.,

$$A^k = A \circ A \circ \dots \circ A.$$

We make the convention that $A^0 = \text{id}$.

We now further restrict our attention to continuous mappings $A \in \text{MP}(\mathbb{R}_+^n)$ that satisfy $A0 = 0$. We have the following characterisation.

Theorem 1 ([11]). *Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. Then the following are equivalent.*

1. For every $x \in \mathbb{R}_+^n$,

$$A^k x \longrightarrow 0 \text{ as } k \rightarrow \infty. \quad (1.1)$$

2. For every $x \in \mathbb{R}_+^n$, $x \neq 0$,

$$Ax \not\leq x.$$

3. Every cycle in the matrix A is a contraction, i.e.,

$$(a_{i_1 i_2} \circ a_{i_2 i_3} \circ \dots \circ a_{i_k i_1})(t) < t$$

for every $t > 0$ and all finite sequences $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$.

4. All minimal cycles in A are contractions, i.e., those that do not contain shorter cycles.

5. For every $b \in \mathbb{R}_+^n$ there is a unique maximal solution $x \in \mathbb{R}_+^n$ to the inequality

$$x \leq Ax \oplus b.$$

Along with an alternative construction of a right eigenvector, a slightly weaker version of this result has been proven in [11, Theorem 6.4], where the functions a_{ij} were assumed to be either strictly increasing or zero. However, the proof is essentially the same in the current framework and thus omitted.

1.4 Main results

Our main technical ingredient for the construction of left and right eigenvectors is the closure of max-preserving maps in the semiring MP.

Lemma 2. *Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. Let any of the conditions 1–5 of Theorem 1 hold. Then the closure of A , given by*

$$A^* x = \bigoplus_{k=0}^{\infty} A^k x \quad (1.2)$$

is a continuous and max-preserving map $A^: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ with $A^*0 = 0$ that satisfies*

$$A^* = \text{id} \oplus AA^* = \text{id} \oplus A^*A. \quad (1.3)$$

Proof. The identities (1.3) follow immediately from writing out (1.2). That A^* is well-defined is mostly a consequence of (1.1), once we note that (1.1) implies that the supremum in (1.2) is a maximum that is attained after a finite number of iterates of A .

The (i, j) th entry of the matrix A^* consists of the supremum over all possible paths from node j to node i in the weighted graph with n vertices and directed edges weighted with the functions a_{ij} . Because any path longer than n edges will contain a cycle, which in turn is a contraction, the infinite

supremum in the definition of A^* , cf. (1.2), is in fact a maximum over at most n powers of A .

Thus A^* is max-preserving. In particular, only a finite number of terms $\|A^k x\|$ can be larger than $\|x\|$ and they depend continuously on $\|x\|$. $\$$

Remark 1. From the proof we see that in fact

$$A^* x = \bigoplus_{k=0}^{n-1} A^k x,$$

a finite maximum of only n vectors instead of a supremum. This will be demonstrated in Section 1.5.

Lemma 3. *Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. Let any of the conditions 1–5 of Theorem 1 hold. Then the closure of A satisfies*

$$A(A^*(x)) = A^*(A(x)) < A^*(x) \quad (1.4)$$

for all $x > 0$.

Proof. First we note that from the definition (1.2) it follows that $A^* A = A A^*$. We have $A^* A \leq A^* A \oplus \text{id} = A^*$ from (1.3), so we only need to show that equality does not hold. To this end assume there is an $x \in \mathbb{R}_+^n$, $x > 0$, with $A^* A x = A^* x$. Denoting $z = A^* x$, we have

$$Az = A A^* x = A^* A x = A^* x = z,$$

which contradicts property 2 of Theorem 1, as $z > x > 0$. Hence no such x can exist, proving that indeed $A A^* x = A^* A x < A^* x$ for all $x > 0$. $\$$

Our main result is the following.

Theorem 2 (left eigenvectors for max-preserving maps). *Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. Let any of the conditions 1–5 of Theorem 1 hold. Then $l: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ given by*

$$x \mapsto \mathbf{1}^T A^*(x) \quad (1.5)$$

is continuous, monotone, satisfies $l(0) = 0$, as well as

1. $l(x) \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$,
2. the left eigenvector inequality

$$lAx \leq lx$$

for all $x \in \mathbb{R}_+^n$, and, moreover, $lAx < lx$ whenever $x \neq 0$.

Proof. That the map l is well defined, continuous, monotone, and satisfies $l(0) = 0$ is an immediate consequence of Lemma 2. Assertion 1 follows from the fact that $A^* \geq \text{id}$. Assertion 2 is a direct consequence of Lemma 3. $\$$

Remark 2. Instead of a summation of the components of $A^*(x)$ in (1.5) we could have taken their maximum instead, at the expense of loosing the strict inequality in Assertion 2 of the theorem. In the context of Section 1.6, this would in general give rise to a weak Lyapunov function, i.e., one that is merely non-increasing along trajectories.

Our notion of left eigenvectors is complemented by right eigenvectors that are given by a similar construction, which, to the best of our knowledge, was first demonstrated in [7]. A different construction is given in [11].

Theorem 3 (right eigenvectors for max-preserving maps [7]). *Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. Let any of the conditions 1–5 of Theorem 1 hold. Then $r: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ given by*

$$t \mapsto A^*(t\mathbf{1}) \quad (1.6)$$

is continuous, monotone, satisfies $r(0) = 0$ as well as

- 1. $r_i(t) \rightarrow \infty$ when $\|t\| \rightarrow \infty$ for every $i = 1, \dots, n$,*
- 2. the right eigenvector inequality*

$$A(r(t)) \leq r(t) \quad (1.7)$$

for all $t \geq 0$, and, moreover, $A(r(t)) < r(t)$ when $t > 0$.

Proof. That r is well defined, continuous, monotone and satisfies $r(0) = 0$ follows again from Lemma 2. Assertion 1 is a consequence of the fact that $A^* \geq \text{id}$, see (1.3), so $r(t) \geq t\mathbf{1}$. Assertion 2 follows from Lemma 3 applied to $x = t\mathbf{1}$. §

Remark 3. In both, Theorem 2 and Theorem 3, instead of the vector $\mathbf{1}$ in the definition of l , respectively, r , any strictly positive vector could have been taken instead.

1.5 Examples

We demonstrate with two examples that the the left and right eigenvectors obtained in the previous section are given by finite expressions, cf. Remark 1, not as limits as the definition in (1.2) might suggest. The examples are borrowed from [12]. To this end we define

$$\mathcal{K}_\infty = \left\{ a: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid a \text{ is continuous, unbounded,} \right. \\ \left. \text{strictly increasing and satisfies } a(0) = 0 \right\},$$

which is the set of homeomorphisms from \mathbb{R}_+ into itself.

First we consider the case $n = 2$. In this case A takes the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{ij} \in (\mathcal{K}_\infty \cup \{0\})$. The associated max-preserving mapping $A: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11}(x_1) \oplus a_{12}(x_2) \\ a_{21}(x_1) \oplus a_{22}(x_2) \end{pmatrix}.$$

The conditions of Theorem 1 are satisfied if and only if

$$a_{11} < \text{id}$$

$$a_{22} < \text{id}$$

and

$$a_{12} \circ a_{21} < \text{id}. \quad (1.8)$$

Note that (1.8) holds if and only if

$$a_{21} \circ a_{12} < \text{id}$$

holds. This can be seen by observing that every \mathcal{K}_∞ function has an inverse which is again a \mathcal{K}_∞ function.

Writing $x = (x_1, x_2)^T$ and under the above assumptions we compute

$$\begin{aligned} A^*(x) &= \bigoplus_{k=0}^{\infty} A^k(x) \\ &= x \oplus Ax = (\text{id}_{\mathbb{R}_+^2} \oplus A)(x) \\ &= \begin{pmatrix} \text{id} & a_{12} \\ a_{21} & \text{id} \end{pmatrix} (x) = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} (x) \end{aligned} \quad (1.9)$$

as already

$$A^2 = \begin{pmatrix} a_{11}^2 \oplus a_{12} \circ a_{21} & a_{12} \circ a_{22} \oplus a_{11} \circ a_{12} \\ a_{21} \circ a_{11} \oplus a_{22} \circ a_{21} & a_{22}^2 \oplus a_{21} \circ a_{12} \end{pmatrix}$$

is component-wise less than the matrix $(\text{id}_{\mathbb{R}_+^2} \oplus A)$ computed above.

From (1.9) we obtain

$$l(x) = x_1 \oplus a_{12}(x_2) + x_2 \oplus a_{21}(x_1)$$

Notably, this function is in general not smooth and neither sum- nor max-separable.

For the case $n = 3$ things are essentially the same.
Starting from

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and under the assumption that all cycles in A are contractions, we can compute A^* simply

$$\begin{aligned} A^* &= \text{id}_{\mathbb{R}_+^3} \oplus A \oplus A^2 \\ &= \begin{pmatrix} \text{id} & a_{12} \oplus a_{13} \circ a_{32} & a_{13} \oplus a_{12} \circ a_{23} \\ a_{21} \oplus a_{23} \circ a_{31} & \text{id} & a_{23} \oplus a_{21} \circ a_{13} \\ a_{31} \oplus a_{32} \circ a_{21} & a_{32} \oplus a_{31} \circ a_{12} & \text{id} \end{pmatrix}, \end{aligned} \quad (1.10)$$

where we note that the simplifications used to obtain (1.10) are possible because all cycles are contractions.

From (1.10) we obtain

$$\begin{aligned} l(x) &= x_1 \oplus (a_{12} \oplus a_{13} \circ a_{32})(x_2) \oplus (a_{13} \oplus a_{12} \circ a_{23})(x_3) \\ &\quad + (a_{21} \oplus a_{23} \circ a_{31})(x_1) \oplus x_2 \oplus (a_{23} \oplus a_{21} \circ a_{13})(x_3) \\ &\quad + (a_{31} \oplus a_{32} \circ a_{21})(x_1) \oplus (a_{32} \oplus a_{31} \circ a_{12})(x_2) \oplus x_3. \end{aligned}$$

1.6 Application

Let $A \in \text{MP}(\mathbb{R}_+^n)$ be continuous and satisfy $A0 = 0$. If $A^k x \rightarrow 0$ for $k \rightarrow \infty$, two types of Lyapunov functions can be defined based on the eigenvectors introduced in the previous section. Let $l: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ and $r: \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ denote the left and right eigenvectors of A , respectively.

Under some additional regularity assumptions, or rather, regularisation of r , a max-separable Lyapunov function $V: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is given by

$$V(x) = \max_i r_i^{-1}(x_i),$$

where r_i denotes the i th component function of r . We refer the interested reader to [7] or to [2] and the references therein for further details.

The left eigenvector l also yields a Lyapunov function $V: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ simply by

$$V(x) = l(x).$$

Theorem 2 establishes that this is indeed a Lyapunov function for the system $x(k+1) = A(x(k))$.

We note that this Lyapunov function is in general neither sum- nor max-separable. However, it has the advantage that no additional regularity has to

be assumed to make the components of the eigenvector invertible and that it can be computed directly from the problem data.

Example 1. Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{7} \\ 2 & \frac{1}{2} & 0 \\ 0 & 3 & \frac{1}{2} \end{pmatrix}$$

where we take the entries as linear functions $t \mapsto a_{ij}t$ and compute Ax in max algebra, making the associated map $A: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ max-preserving.

There are five cycles in this matrix. Three of them are “self-loops” of weight $1/2$. The other two are from node 1 to 2 with weight 2 and back to node 1 with weight $1/3$, as well as from 1 to 2 with weight 2, from there to 3 with weight 3 and back to 1 with weight $1/7$. All of the loop-weights (products) are less than one, so this matrix satisfies the equivalent conditions of Theorem 1.

A simple computation yields

$$A^* = \begin{pmatrix} 1 & 2 & 6 \\ \frac{3}{7} & 1 & 3 \\ \frac{1}{7} & \frac{2}{7} & 1 \end{pmatrix}.$$

From here we obtain $l(x) = \max\{x_1, 2x_2, 6x_3\} + \max\{\frac{3}{7}x_1, x_2, 3x_3\} + \max\{\frac{1}{7}x_1, \frac{2}{7}x_2, x_3\}$ and we verify that for $x > 0$ the expression $l(Ax) = \max\{\frac{6}{7}x_1, 2x_2, 6x_3\} + \max\{\frac{3}{7}x_1, \frac{6}{7}x_2, 3x_3\} + \max\{\frac{1}{7}x_1, \frac{2}{7}x_2, \frac{6}{7}x_3\}$ is indeed smaller.

1.7 Conclusion

For max-preserving maps A on \mathbb{R}_+^n we have shown that left and right eigenvectors can be defined in a natural sense based on the closure of the map A , extending the classical Perron-Frobenius theory appropriately to nonlinear dominant eigenvalues. In this work the dominant eigenvalue was assumed to be less than the identity, but via suitable scaling this could be extended to more general scenarios.

Our results have been presented on \mathbb{R}_+^n , however, an extension to join-morphisms acting on Banach lattices is a natural next step.

The construction of left-eigenvectors and corresponding Lyapunov functions for general monotone systems that are not generated by elements of a semiring remains a challenge.

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